university of groningen

faculty of science and engineering

Control Engineering for BME

Midterm Exam

Academic year 2021/2022 1A

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The exam is designed to last two hours; from 6:30-8:30. *The exam will cover the topics from lecture 1 to lecture 9.* Please, read carefully the following information.

- Please write completely your name, student ID number and study program.
- **This take-home exam is admissible IF AND ONLY IF you have submitted the signed statement for participating in the take-home test as communicated before with the time-stamp before 6:29, October 8th, 2021.** You can find the student statement document in Nestor, under the section "Midterm".
- The exam contains four questions with compulsory sub-questions and an optional sub-question for which extra points can be granted (if correct).
- For every question, write your answer neatly on blank papers and write down your name and student number on the top of each page.
- This is an OPEN book take-home exam and you are allowed to use calculator or computer. Please write down your answer clearly and with proper argumentation/reasoning whenever needed. **Providing only the final answers without proper argumentation is NOT acceptable and will NOT be graded**[1].
- Please, write your answer using a pen, not a pencil.
- If you are unclear about a specific problem, you can make your own assumptions. Describe your assumptions at the beginning of your answer. **Keep in mind that if the assumptions are no correct, your solution will not be ether.**
- **Whenever we think is appropriate, a follow-up ORAL examination to suspected cases will be arranged before the final grade is determined. In this case, the follow-up oral examination will be based on the questions of this exam and the final grade will be based on the same weighting factor as before where the adjusted grade from the oral examination will be used instead that replaces the final exam grade.**

[1] Concepts over computations, but both are important.

- After the exam is finished, you scan the answer of every question and assign it with the filename: $sX_1X_2X_3X_4X_5X_6X_7\_\_Y_1Y_2.pdf$, where $X_1X_2X_3X_4X_5X_6X_7$ is your student number and Y_1Y_2 is the question number. **That is, ONE question PER pdf file; four pdf files in total.** For instance, if your student number is $s1234567$, then the file for exercise 3 is: $s1234567$_03.pdf.

- After the exam is finished, you have 30 minutes to scan your answers and submit each pdf file via a Nestor Dropbox link. The time-stamp of your submitted answers in the Nestor Dropbox determines the admissibility of your exam answers. If you encounter a problem during the scanning or submission process, let us know as soon as possible via the Blackboard Collaborate tools or via e-mail r.reyes.baez@rug.nl. We will not process your answers if the time-stamp of the documents in Nestor Dropbox is after 21:01, October 9th, 2021. Those students who have the right for extra time will be able to submit their answer to Nestor Dropbox until 21:31, October 8th, 2021.

For the grader only

	Exercise 1	**Exercise 2**	**Exercise 3**	**Exercise 4**
Points				
Bonus				

Exercise 1:

Consider the case of modeling the motion of the private NS train of King Willem Alexander of the Netherlands. This train consist only of two cars of masses m_1 and m_2, interconnected by a spring with constant k_{12} and a damper with constant b_{12}; see the engineering diagram of Figure 1. The train engine exerts a force F. Note that the friction forces between the cars' wheels and tracks are not neglected; this friction forces have coefficients b_1 and b_2, respectively.

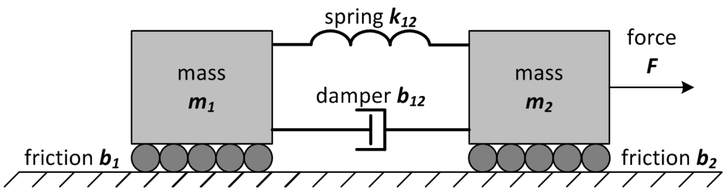

Figure 1: Two cars interconnected by a spring and a damper.

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As control engineer, you decide that a good modeling approach of the friction forces is to interpreting them as damping forces that act on both cars. Then, you propose the following schematic that models the main forces acting in the overall train.

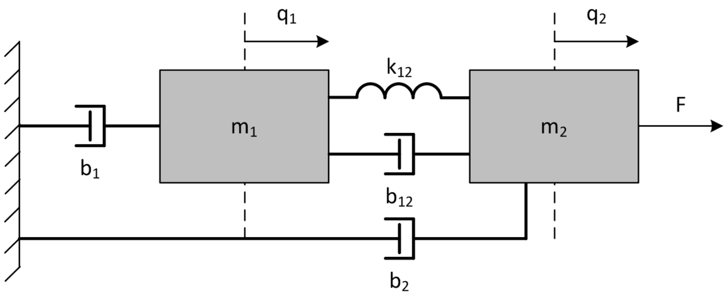

Figure 2: Schematic diagram of the mass-spring-damper system

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The equations of motion of this system are

$$\begin{aligned} m_1\ddot{q}_1 + b_{12}(\dot{q}_1 - \dot{q}_2) + b_1\dot{q}_1 + k_{12}(q_1 - q_2) &= 0 \\ m_2\ddot{q}_2 + b_{12}(\dot{q}_2 - \dot{q}_1) + b_2\dot{q}_2 + k_{12}(q_2 - q_1) &= F. \end{aligned} \tag{1}$$

Show that these are indeed the equation of motion via

1. **[1 pts]** The Newton's laws method, that is, $m\ddot{q}_i = \sum F_k$, with F_k the k-th force and $i = 1, 2$.

2. **[0.5 pts]** The Euler-Lagrange method, with the kinetic and potential energies, respectively, given by

$$K(\dot{q}_1, \dot{q}_2) = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2, \qquad P(q_1, q_2) = \frac{1}{2}k_{12}(q_2 - q_1)^2, \tag{2}$$

   and Rayleigh dissipation function

$$R(\dot{q}_1, \dot{q}_2) = \frac{1}{2}b_{12}(\dot{q}_2 - \dot{q}_1)^2 + \frac{1}{2}b_1\dot{q}_1^2 + \frac{1}{2}b_2\dot{q}_2^2. \tag{3}$$

3. (For bonus **[0.25 pts]**) Show that the equations of motion in (1) can be rewritten as
$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + Bq = M\tau \tag{4}$$
where $q = [q_1, q_2]^\top$, $C(q,\dot{q}), B$ and M are matrices of appropriate dimensions, and $\tau = [0, F]^\top$.

Solution:

1. The forces acting on mass m_1 are the following: Thus, using the second Newton's

Forces acting on each mass

m_1	$-k_{12}q_1$	$k_{12}q_2$	$-b_{12}\dot{q}_1$	$b_{12}\dot{q}_2$	$-b_1\dot{q}_1$	
m_2	F	$-k_{12}q_2$	$k_{12}q_1$	$-b_{12}\dot{q}_2$	$b_{12}\dot{q}_1$	$-b_2\dot{q}_2$

law for each mass, we have
$$\begin{aligned} m_1\ddot{q}_1 &= -k_{12}q_1 + k_{12}q_2 - b_{12}\dot{q}_1 + b_{12}\dot{q}_2 - b_1\dot{q}_1, \\ m_2\ddot{q}_2 &= F - k_{12}q_2 + k_{12}q_1 - b_{12}\dot{q}_2 + b_{12}\dot{q}_1 - b_2\dot{q}_2. \end{aligned} \tag{5}$$
Grouping with respect the constants k_{12} and b_{12}, we get
$$\begin{aligned} m_1\ddot{q}_1 &= -k_{12}(q_1 - q_2) - b_{12}(\dot{q}_1 - \dot{q}_2) - b_1\dot{q}_1, \\ m_2\ddot{q}_2 &= F - k_{12}(q_2 - q_1) - b_{12}(\dot{q}_2 - \dot{q}_1) - b_2\dot{q}_2. \end{aligned} \tag{6}$$
Arranging all the internal forces the left-hand side, and the external forces to the left-hand side, ones gets
$$\begin{aligned} m_1\ddot{q}_1 + k_{12}(q_1 - q_2) + b_{12}(\dot{q}_1 - \dot{q}_2) + b_1\dot{q}_1 &= 0, \\ m_2\ddot{q}_2 + k_{12}(q_2 - q_1) + b_{12}(\dot{q}_2 - \dot{q}_1) + b_2\dot{q}_2 &= F. \end{aligned} \tag{7}$$
which is indeed the equations of motion in (1).

2. The Lagrangian function is[2]
$$\begin{aligned} L(q_1, q_2, \dot{q}_1, \dot{q}_2) &= K(\dot{q}_1, \dot{q}_2) - P(q_1, q_2), \\ &= \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2 - \frac{1}{2}k_{12}(q_2 - q_1)^2. \end{aligned} \tag{8}$$
We compute the Euler-Lagrange equations with dissipation for q_1 and q_2, that is,
$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{q}_k}(q_k, \dot{q}_k)\right) - \frac{\partial L}{\partial q_k}(q_k, \dot{q}_k) = \tau_k - \frac{\partial R}{\partial \dot{q}_k}(q_k, \dot{q}_k), \quad k \in \{1, 2\}. \tag{9}$$
For q_1 (or $k = 1$):
$$\begin{aligned} &\frac{\partial L}{\partial \dot{q}_1}(q_1, \dot{q}_1) = m_1\dot{q}_1 \Rightarrow \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{q}_1}(q_1, \dot{q}_1)\right) = m_1\ddot{q}_1; \\ &\frac{\partial L}{\partial q_1}(q_1, \dot{q}_1) = k_{12}(q_2 - q_1); \\ &\frac{\partial R}{\partial \dot{q}_k}(q_k, \dot{q}_k) = -b_{12}(\dot{q}_2 - \dot{q}_1) + b_1\dot{q}_1 \end{aligned} \tag{10}$$

[2]If you computed everything for $k_{12} = 1$ in the potential energy, we are taking it as correct.

Plugging everything into (9), we have

$$m_1\ddot{q}_1 - k_{12}(q_2 - q_1) = \tau_1 + b_{12}(\dot{q}_2 - \dot{q}_1) - b_1\dot{q}_1. \tag{11}$$

The external force is $\tau_1 = 0$ for the mass m_1. Thus, reordering all the internal forces to the left-hand side and the external forces to the left,

$$m_1\ddot{q}_1 + b_{12}(\dot{q}_1 - \dot{q}_2) + b_1\dot{q}_1 + k_{12}(q_1 - q_2) = 0, \tag{12}$$

which is the first equation in (1).
For q_2 (or $k = 2$) in (9):

$$\begin{aligned}
&\frac{\partial L}{\partial \dot{q}_1}(q_1, \dot{q}_1) = m_2\dot{q}_2 \Rightarrow \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{q}_1}(q_1, \dot{q}_1)\right) = m_1\ddot{q}_1;\\
&\frac{\partial L}{\partial q_1}(q_1, \dot{q}_1) = -k_{12}(q_2 - q_1);\\
&\frac{\partial R}{\partial \dot{q}_k}(q_k, \dot{q}_k) = b_{12}(\dot{q}_2 - \dot{q}_1) + b_2\dot{q}_2
\end{aligned} \tag{13}$$

Plugging everything into (9), we have

$$m_2\ddot{q}_1 + k_{12}(q_2 - q_1) = \tau_1 - b_{12}(\dot{q}_2 - \dot{q}_1) - b_2\dot{q}_2. \tag{14}$$

The external force is $\tau_1 = F$ for the mass m_2. Thus, reordering all the internal forces to the left-hand side and the external forces to the left,

$$m_2\ddot{q}_1 + b_{12}(\dot{q}_2 - \dot{q}_1) + b_2\dot{q}_2 + k_{12}(q_2 - q_1) = F, \tag{15}$$

which is the second equation in (1).

3. Grouping in matrix the equations of motion in (1), one gets the following

$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{D(q)} \underbrace{\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}}_{\ddot{q}} + \underbrace{\begin{bmatrix} b_{12} + b_1 & -b_{12} \\ -b_{12} & b_{12} + b_2 \end{bmatrix}}_{C(q,\dot{q})} \underbrace{\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}}_{\dot{q}} + \underbrace{\begin{bmatrix} k_{12} & -k_{12} \\ -k_{12} & k_{12} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}}_{q} = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{M} F. \tag{16}$$

Exercise 2:

The metabolism of alcohol in the body can be modeled by the normalized nonlinear compartmental model

$$\begin{aligned}
\dot{x}_1 &= (x_2 - x_1) + u,\\
\dot{x}_2 &= (x_1 - x_2) - \frac{x_2}{1 + x_2} + u,
\end{aligned} \tag{17}$$

where

- $x_1, x_2 \in \mathbb{R}$ are the concentrations of alcohol in the compartment.
- $u \in \mathbb{R}$ is the intravenous and gastrointestinal injection rate.

Answer the following questions:

1. **[1 pts]** Given a constant input $\overline{u}$, with $\overline{u}$ positive, determine the operation point $\overline{x} = [\overline{x}_1, \overline{x}_2]^\top$ of the system. State a condition on $\overline{u}$ that guarantees the equilibrium $\overline{x}$ to have both positive components.

2. [**1 pts**] Linearize the nonlinear dynamics of the compartmental model in (17) around the operation point

$$\overline{x} = \begin{bmatrix} \frac{5}{4} \\ 1 \end{bmatrix}, \qquad \overline{u} = \frac{1}{4}. \tag{18}$$

That is, determine the matrices A, B of the Jacobian linear approximation of (17) in

$$\delta\dot{x} = A\delta x + B\delta u \tag{19}$$

Hint: compute directly the matrices A,B, you do not need to do the whole process as in the lectures.

3. [**1 pts**] For the linearized system obtained in the previous sub-problem 2.2,

$$\delta\dot{x} = A\delta x + B\delta u. \tag{20}$$

Analyze the stability of the equilibrium point of the open-loop system. Is it asymptotically stable, stable or unstable? What is the expected behavior of the operation pair in (18) for the nonlinear model in (17)?
Hint: If and only if you did not determine the matrices A, B in sub-problem 2.2, then use the following ones

$$A = \begin{bmatrix} -3 & 2 \\ 2 & -2 \end{bmatrix} x, \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{21}$$

4. (For bonus [**0.5 pts**]) Let the output be

$$\delta y = \delta x_1. \tag{22}$$

Compute, if it exist, the steady-state input/output response of the linearized system to the step function $1(t)$. That is, the steady-state output response of the linearized system when $\delta x(0) = x(0)$ and $\delta u = 1$, for all $t \geq 0$.

Solution:

1. Note that from the instructions, the nominal constant input $\overline{u}$ is given, which satisfies $\overline{u} > 0$; thus, the operational equilibrium $(\overline{x}_1, \overline{x}_2)$ will be a function of the given input $\overline{u}$. It follows that the operation points $(\overline{x}, \overline{u})$ of the nonlinear compartmental system in (17) are the solution to the set of algebraic equations

$$\begin{aligned} 0 &= (\overline{x}_2 - \overline{x}_1) + \overline{u}, \\ 0 &= (\overline{x}_1 - \overline{x}_2) - \frac{\overline{x}_2}{1 + \overline{x}_2} + \overline{u}. \end{aligned} \tag{23}$$

From the first equation, we have that $(\overline{x}_2 - \overline{x}_1) = -\overline{u}$ or $(\overline{x}_1 - \overline{x}_2) = \overline{u}$. Substitution of this into the second equation yields

$$0 = \overline{u} - \frac{\overline{x}_2}{1 + \overline{x}_2} + \overline{u} \iff \frac{\overline{x}_2}{1 + \overline{x}_2} = 2\overline{u}, \iff \overline{x}_2 = 2\overline{u}(1 + \overline{x}_2). \tag{24}$$

Solving for $\overline{x}_2$ in the last equation, one gets (after some minor computations)

$$\overline{x}_2 = \frac{2\overline{u}}{1 - 2\overline{u}}. \tag{25}$$

To get $\overline{x}_1$, substitute (25) into the first equation of (23), and solve for $\overline{x}_1$

$$
\begin{aligned}
0 &= (\overline{x}_2 - \overline{x}_1) + \overline{u}, \\
\overline{x}_1 &= \frac{2\overline{u}}{1-2\overline{u}} + \overline{u} = \frac{3\overline{u} - 2\overline{u}^2}{1-2\overline{u}}.
\end{aligned} \tag{26}
$$

Hence, the operation points of the system (17) are

$$
(\overline{x}, \overline{u}) = \left(\begin{bmatrix} \frac{3\overline{u}-2\overline{u}^2}{1-2\overline{u}} \\ \frac{2\overline{u}}{1-2\overline{u}} \end{bmatrix}, \overline{u} \right), \quad \text{for} \quad \overline{u} > 0. \tag{27}
$$

To determine conditions on $\overline{u} > 0$ such that $\overline{x}_1 > 0$ and $\overline{x}_2 > 0$, we solve the following set of inequalities

$$
\overline{x}_1 = \frac{3\overline{u} - 2\overline{u}^2}{1-2\overline{u}} > 0 \quad \text{and} \quad \overline{x}_2 = \frac{2\overline{u}}{1-2\overline{u}} > 0, \tag{28}
$$

Clearly, $\overline{x}_1 > 0$ and $\overline{x}_2 > 0$ if the corresponding numerators and denominators are positive; in other words, the following conditions must be simultaneously satisfied

$$
3\overline{u} - 2\overline{u}^2 > 0 \quad \text{and} \quad 1 - 2\overline{u} > 0 \quad \text{and} \quad 2\overline{u} > 0 \quad \text{and} \quad 1 - 2\overline{u} > 0, \tag{29}
$$

or, equivalently,

$$
3 > 2\overline{u} \quad \text{and} \quad 1 > 2\overline{u} \quad \text{and} \quad \overline{u} > 0 \quad \text{and} \quad 1 > 2\overline{u}, \tag{30}
$$

where the first and third inequalities follow by virtue of $\overline{u} > 0$, and the second and fourth are the same condition. Then, $\overline{u}$ must satisfy

$$
\frac{3}{2} > \overline{u} \quad \text{and} \quad \frac{1}{2} > \overline{u}. \tag{31}
$$

Therefore, $\overline{x}_1 > 0$ and $\overline{x}_2 > 0$ if $\overline{u} < 1/2$.

2. The operation point for $\overline{u} = 1/4$ is given by

$$
\overline{x} = \begin{bmatrix} \frac{5}{4} \\ 1 \end{bmatrix}, \qquad \overline{u} = \frac{1}{4}. \tag{32}
$$

To find the Jacobian linear approximation of the nonlinear compartmental model in (17), we simply compute the pair (A, B), and write the linear dynamics of the incremental state $\delta x = x - \overline{x}$ with incremental input $\delta u = u - \overline{u}$. Then,

$$
\begin{aligned}
A &= \begin{bmatrix} \frac{\partial}{\partial x_1}\left[(x_2 - x_1) + u\right] & \frac{\partial}{\partial x_2}\left[(x_2 - x_1) + u\right] \\ \frac{\partial}{\partial x_1}\left[(x_1 - x_2) - \frac{x_2}{1+x_2} + u\right] & \frac{\partial}{\partial x_2}\left[(x_1 - x_2) - \frac{x_2}{1+x_2} + u\right] \end{bmatrix}_{(\overline{x},\overline{u})} \\
&= \begin{bmatrix} -1 & 1 \\ 1 & -1 - \frac{(1+x_2) - x_2}{(1+x_2)^2} \end{bmatrix}_{(\overline{x},\overline{u})} = \begin{bmatrix} -1 & 1 \\ 1 & -\frac{5}{4} \end{bmatrix}. \\
B &= \begin{bmatrix} \frac{\partial}{\partial u}\left[(x_2 - x_1) + u\right] \\ \frac{\partial}{\partial x_1}\left[(x_1 - x_2) - \frac{x_2}{1+x_2} + u\right] \end{bmatrix}_{(\overline{x},\overline{u})} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{(\overline{x},\overline{u})} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\end{aligned} \tag{33}
$$

It follows that the Jacobian linear approximation of (17) around the operation point (32) is the following LTI system

$$\delta\dot{x} = \underbrace{\begin{bmatrix} -1 & 1 \\ 1 & -\frac{5}{4} \end{bmatrix}}_{A} \delta x + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{B} \delta u. \tag{34}$$

To analyze the stability of the open-loop system, let us compute the characteristic polynomial of A for $\delta u = 0$.

$$\begin{aligned} p(s) &= \det\left\{ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 1 & -\frac{5}{4} \end{bmatrix} \right\} \\ &= \det \begin{bmatrix} s+1 & -1 \\ -1 & s+\frac{5}{4} \end{bmatrix}, \\ &= s^2 + \frac{9}{4}s + \frac{1}{4}. \end{aligned} \tag{35}$$

Instead of computing the eigenvalues directly, let us use the Routh-Hurwitz criterion

s^2	1	$\frac{1}{4}$
s^1	$\frac{9}{4}$	0
s^0	$\frac{1}{4}$	

Since no sign changes in the first column of the Routh's array, we conclude that the two eigenvalues lie in the left half complex plane (LHP), and the origin is an asymptotically stable. Therefore, by the linearization Lyapunov theorem we conclude that the equilibrium point $\overline{x}$ is asymptotically stable for the nonlinar system.

3. For the output $\delta y = \delta x_1$, the matrix C is given by $C = [1\ 0]$. From the instructions in Exercise 4 in (64), we know that the input/output response to the step function for $x(0) = 0$ is $r(t) = CA^{-1}(e^{At} - I)B$. Since A is Hurwitz, the following holds

$$y_{ss} = \lim_{t\to\infty} CA^{-1}(e^{At} - I)B = -CA^{-1}B = -\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -5 & -4 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 9. \tag{36}$$

Exercise 3:

Consider the linear system

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u, \tag{37}$$

where $x = [x_1, x_2]^\top \in \mathbb{R}^2$ is the state vector, $u \in \mathbb{R}$ is the control input, and $y \in \mathbb{R}$ is the measured output, b_1, b_2 are uncertain parameters.

1. **[1 pts]** Determine condition on the parameters b_1, b_2 such that the system is reachable using the reachability matrix.

2. **[2 pts]** Consider the nominal values $b_1 = 0$ and $b_2 = 1$. Determine the gain matrix K such that the closed-loop system has an input/output response to the step input $u = 1(t)$ with an overshoot percentage $M_p(\%OS) = 30\%$ and peak time of $T_p = 127$ sec. Where are the eigenvalues of the closed-loop matrix $A_{\text{cl}} = A - BK$ located?

3. **[1 pts]** Close the loop of system of system (37) with the full state feedback control $u = -Kx$ that you determined in sub-problem 3.2, with $b_2 = 1$ and determine a condition on b_1 such that the closed-loop system remains asymptotically stable. *Hint: If you didnt answer sub-problem 3.2, use* $K = [9\ 4]$.

4. (For bonus **[0.5 pts]**) Consider the system in (37) with $b_1 = 0$ and $b_2 = 1$. Suppose that one adds an integral action to the full state feedback $u = -[9\ 4]x$, that is,

$$u = -Kx + \int_0^t x_1(\tau)d\tau. \tag{38}$$

Write the state equations of system (37) in closed-loop with the controller (38).

Solution:

1. The reachability matrix for the LTI system in (37) is

$$W_r = \begin{bmatrix} B & \vdots & AB \end{bmatrix} = \begin{bmatrix} b_1 & (b_1 + b_2) \\ b_2 & 0 \end{bmatrix} \tag{39}$$

We know that the pair (A, B) is reachable if and only if W_r is full rank. Since the matrix W_r is 2×2, the full rank condition is equivalent to prove that $\det(W_r) \neq 0$. Thus, the following must hold

$$\begin{aligned} \det(W_r) \neq 0 &\iff -b_1 b_2 - b_2^2 \neq 0, \\ \det(W_r) \neq 0 &\iff -b_1 b_2 \neq b_2^2, \\ \det(W_r) \neq 0 &\iff -b_1 \neq b_2, \quad \text{for} \quad b_2 \neq 0, \end{aligned} \tag{40}$$

Hence, the pair (A, B) is reachable if and only if $-b_1 \neq b_2$ and $b_2 \neq 0$.

2. For $b_1 = 0$ and $b_2 = 1$, we get the following system from (37)

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u. \tag{41}$$

From sub-problem 3.1, we already know that the pair (A, B) is reachable, otherwise we would have needed to perform an stabilizability test. The open-loop characteristic polynomial is

$$p(s) = s^2 - s = s^2 + \underbrace{(-1)}_{a_1} s + \underbrace{0}_{a_2}. \tag{42}$$

Clearly, the necessary condition for stability is not met, and the open-loop system's equilibrium point $x = 0$ is unstable.

To stabilize the equilibrium point and impose a closed-loop input/output time response to step function with the above desired performance in terms of the percentage of overshoot $\%OS$ and the peak time T_p, one can define the target closed-loop characteristic polynomial

$$p_{\text{tg}}(s) = s^2 + 2\omega_n \zeta s + \omega_n^2. \tag{43}$$

where the damping ratio ζ and the natural frequency ω_n are related to the $\%OS$ and T_p by means of the following formulas[3]:

$$\begin{aligned}
\zeta &= \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}} = \frac{-\ln(0.30)}{\sqrt{\pi^2 + \ln^2(0.30)}} = 0.36, \\
\omega_n &= \frac{\pi}{T_p\sqrt{1-\zeta^2}} = \frac{\pi}{127\sqrt{1-0.36^2}} = 0.026.
\end{aligned} \tag{44}$$

After substitution of the numerical values of ζ and ω, the target closed-loop characteristic polynomial becomes

$$p_{\text{tg}}(s) = s^2 + \underbrace{0.0187}_{\alpha_1}s + \underbrace{0.0007}_{\alpha_2}, \tag{45}$$

that has the target eigenvalues located at -0.0094 ± 0.0243.

The control design goal is to match the target characteristic polynomial in (43) with the closed-loop one via the gain matrix $K = [k_1\ k_2]$. That is, the characteristic polynomial of the closed-loop system's matrix,

$$A_{\text{cl}} = A - BK = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -k_1 & -k_2 \end{bmatrix}, \tag{46}$$

given by

$$p_{\text{cl}}(s) = \det \begin{bmatrix} s-1 & -1 \\ k_1 & s+k_2 \end{bmatrix} = s^2 + (k_2 - 1)s + (k_1 - k_2). \tag{47}$$

To match the closed-loop characteristic polynomial in (47) with the target one in (45), we need to impose the same coefficients via the gains k_1 and k_2 as follows

$$k_2 - 1 = \alpha_1, \qquad k_1 - k_2 = \alpha_2, \tag{48}$$

or

$$k_2 = \alpha_1 + 1, \qquad k_1 = \alpha_2 + \alpha_1 + 1. \tag{49}$$

Substitution of the numerical values yields the numerical values of the gains

$$k_2 = 1.0187, \qquad k_1 = 1.0194. \tag{50}$$

3. For this item, with $b_2 = 1$, from (37) we get the system

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} b_1 \\ 1 \end{bmatrix} u. \tag{51}$$

With the gain matrix $[k_1\ k_2] = [1.0194 \quad 1.0187]$ of sub-problem 3.2, we get the following closed-loop matrix

$$A_{\text{cl}} = A - BK = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} b_1 \\ 1 \end{bmatrix} [k_1\ k_2] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} b_1k_1 & b_1k_2 \\ k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 1-b_1k_1 & 1-b_1k_2 \\ -k_1 & -k_2 \end{bmatrix}. \tag{52}$$

The closed-loop characteristic polynomial in this case is

$$p_{\text{cl}}(s) = \det \begin{bmatrix} s-(1-b_1k_1) & -(1-b_1k_2) \\ k_1 & s+k_2 \end{bmatrix} = s^2 + (k_2 - 1 + b_1k_1)s + (k_1 - k_2). \tag{53}$$

Let us use the Routh-Hurwitz criterion to determine conditions on b_1 such that (53) is a strictly stable polynomial. Thus, the Routh's table is

[3]See exercise 7 of the tutorial of week 29.

s^2	1	$(k_1 - k_2)$
s^1	$(k_2 - 1 + b_1 k_1)$	0
s^0	$(k_1 - k_2)$	

From the numerical values of k_1, k_2 in (50), clearly $(k_1 - k_2) > 0$. Hence, the constant b_1 must satisfy

$$k_2 - 1 + b_1 k_1 > 0 \iff b_1 > \frac{1 - k_2}{k_1} = -0.0184. \tag{54}$$

4. In this case, for $b_1 = 0, b_2 = 1$, we use the following system

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \tag{55}$$

with the controller given by[4]

$$u = -Kx + \int_0^t x_1(\tau) d\tau. \tag{56}$$

The integral of a state defines another state, thus, the closed-loop system should have three states. Let x_u be the state of the controller. Thus, using this new state, by the fundamental theorem of calculus, the controller with integral action (56) can be equivalently rewritten as

$$\begin{aligned} \dot{x}_u &= x_1 \\ u &= -Kx + x_u. \end{aligned} \tag{57}$$

Substitution in (55) yields

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(\begin{bmatrix} -9 & -4 \end{bmatrix} x + x_u \right), \\ \dot{x}_u &= x_1, \end{aligned} \tag{58}$$

or

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & 1 \\ -9 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_u, \\ \dot{x}_u &= x_1. \end{aligned} \tag{59}$$

Arranging everything in matrix form, we get the following

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_u \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -9 & -4 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_u \end{bmatrix}. \tag{60}$$

Exercise 4:

Consider the system defined by

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx \end{aligned} \tag{61}$$

[4] This exercise is similar to Exercise 7.2 of tutorial 2

1. **[1 pts]** Show that the input/output response of to the ramp function defined as

$$u(t) = t, \qquad t \geq 0. \tag{62}$$

is given by

$$y(t) = Ce^{At}x(0) + C\left[A^{-2}(e^{At} - I) - A^{-1}t\right]B. \tag{63}$$

Hint: Consider that $\int_0^t e^{-A\tau} d\tau = (I - e^{-At})A^{-1} = A^{-1}(I - e^{-At})$.

2. **[0.5 pts]** What is the steady-state output response y_{ss} of sub-problem 4.1

3. (For bonus **[0.25 pts]**) Suppose that the initial conditions are zero, i.e., $x(0) = 0$. Show that the output response to the ramp in (63) is the integral of the input/output response to the unitary step $1(t)$ given by

$$r(t) = CA^{-1}(e^{At} - I)B. \tag{64}$$

Solution:

1. **Solution:**
We need to compute the explicit solution to the state equation, i.e.,

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau) \cdot 1(\tau)d\tau, \tag{65}$$

for the ramp input $u(t) = t1(t)$, that is,

$$\begin{aligned} x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}B\tau d\tau, \\ &= e^{At}x(0) + e^{At}\int_0^t e^{-A\tau}\tau d\tau B. \end{aligned} \tag{66}$$

Using integration by parts for the indefinite integral

$$\int e^{-A\tau}\tau d\tau, \tag{67}$$

consider the following definitions:

$$\begin{aligned} \mu &= \tau \qquad & d\nu &= e^{-At}d\tau, \\ d\mu &= d\tau \qquad & \nu &= \int e^{-At}d\tau. \end{aligned} \tag{68}$$

Then,

$$\int e^{-A\tau}\tau d\tau = \mu\nu - \int \nu d\mu \tag{69}$$

Substitution of the hint in (69) yields

$$\begin{aligned} \int e^{-A\tau}\tau d\tau &= \tau\left[I - e^{-A\tau}\right]A^{-1} - \int\left[I - e^{-A\tau}\right]A^{-1}d\tau, \\ &= \tau\left[I - e^{-A\tau}\right]A^{-1} - \int A^{-1}d\tau + \int e^{-A\tau}d\tau A^{-1}, \\ &= \tau A^{-1} - \tau e^{-A\tau}A^{-1} - A^{-1}\tau + \left[I - e^{-A\tau}\right]A^{-2}, \\ &= -\tau e^{-A\tau}A^{-1} + \left[I - e^{-A\tau}\right]A^{-2}. \end{aligned} \tag{70}$$

Thus, we can plug this last expression into (66) as follows

$$\begin{aligned}
x(t) &= e^{At}x(0) + e^{At}\int_0^t e^{-A\tau}\tau d\tau B, \\
&= e^{At}x(0) + e^{At}\left[-\tau e^{-A\tau}A^{-1} + \left[I - e^{-A\tau}\right]A^{-2}\right]|_{\tau=0}^{\tau=t}B, \\
&= e^{At}x(0) + e^{At}\left[-te^{-At}A^{-1} + \left[I - e^{-At}\right]A^{-2}\right]B, \\
&= e^{At}x(0) + \left[-te^{At}e^{-At}A^{-1} + \left[e^{At} - e^{At}e^{-At}\right]A^{-2}\right]B, \\
&= e^{At}x(0) + \left[-tA^{-1} + \left[e^{At} - I\right]A^{-2}\right]B,
\end{aligned} \tag{71}$$

From (**??**), we know that $(I - e^{-A\tau})A^{-1} = A^{-1}(I - e^{-A\tau})$. This in turn implies that

$$\begin{aligned}
x(t) &= e^{At}x(0) + \left[-tA^{-1} + \left[e^{At} - I\right]A^{-2}\right]B, \\
&= e^{At}x(0) + \left[-tA^{-1} + A^{-1}\left[e^{At} - I\right]A^{-1}\right]B, \\
&= e^{At}x(0) + \left[-tA^{-1} + A^{-2}\left[e^{At} - I\right]\right]B.
\end{aligned} \tag{72}$$

Therefore, the I/O response to the ramp function is given by

$$y(t) = Ce^{At}x(0) + C\left[A^{-2}(e^{At} - I) - A^{-1}t\right]B. \tag{73}$$

2. A necessary condition for the existence of the steady-state is that A is Hurwitz. Thus, if A is Hurwitz, $e^{At} \to 0$ as $t \to \infty$, and all the terms multiplying the exponential matrix in (73) will vanish. Hence,

$$y_{ss} = \lim_{t\to\infty}\left(Ce^{At}x(0) + C\left[A^{-2}(e^{At} - I) - A^{-1}t\right]B\right) = -CA^{-1}Bt. \tag{74}$$

3. In this exercise, we can compute directly the integral of the I/O response to the step in (64); or, due to the fundamental theorem of calculus, to differentiate (73) with respect to the time, and showing that the result is (64). We will proceed as in the later. Take $x(0) = 0$ as indicated in the exercise.

$$\begin{aligned}
\frac{dy}{dt}(t) &= \frac{d}{dt}\left(C\left[A^{-2}(e^{At} - I) - A^{-1}t\right]B\right), \\
&= C\left[A^{-2}(Ae^{At}) - A^{-1}\right]B, \\
&= C\left[A^{-1}e^{At} - A^{-1}\right]B, \\
&= CA^{-1}\left[e^{At} - I\right]B, \\
&= r(t).
\end{aligned} \tag{75}$$